Edge geo chromatic number of a graph

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Abstract The Edge Geodetic Number is also known as the smallest size among all sets of edges that make up Edge Geodetic sets and is represented by the symbol \(g(G)\). Assuming \(G\) is a finite connected graph, we can define a set \(S \subseteq V(G)\) as an Edge Geodetic set if, for any pair of vertices \(u\) and \(v\) in set \(S\), every edge in \(G\) lies on the shortest path between \(u\) and \(v\). A set \(C \subseteq E(G)\) is referred to as a chromatic index set when it contains all of the proper \(k\)-edge colors of \(G\). The chromatic index is determined by the set’s lowest cardinality among its edges. The “Chromatic Index Number” is a term used to refer to the collection of these minimum cardinality chromatic index values and is represented by the symbol \(\chi’(G)\). By combining an Edge Geodetic set with a Chromatic Index set, the Edge Geo Chromatic set of \(G\) is created, providing an innovative concept. The Edge Geo Chromatic Number, abbreviated as \(X\), is the least cardinality among all Edge Geo Chromatic sets of \(G\). In essence, an Edge Geodetic set and a Chromatic Index set combine to form an Edge Geo Chromatic graph. We talk about an edge geo linked graph with \(n\) orders. For a number of common graphs, we have computed the Edge Geo Chromatic Number and established its bounds. Furthermore, we have shown that when \(G\) represents a graph of diameter \(d\), then \(X(G) \leq d^2+2\).

Keywords: geodetic number, edge geodetic number, chromatic index number, edge geo chromatic number

1. Introduction

As defined in prior works (Bondy et al., 1976; Buckley et al., 1990), \(p\) and \(q\) represent \(G\)’s order and size, referring to the edge set \(E(G)\) and vertex set \(V(G)\) of \(G\). The shortest distance between any pair of vertices within \(V(G)\) corresponds to the minimum length of a path from \(v_1\) to \(v_2\) in \(G\). A path spanning \(d_G(v_1, v_2)\) between vertices \(v_1\) and \(v_2\) is termed a geodesic within \(G\). We use the term \(l_G(v_1, v_2)\) to describe the set of vertices contained within specific \(v_1\)-\(v_2\) geodesics of \(G\). The geodesic \(v_1\)-\(v_2\) is considered to encompass vertex \(C\) if \(C\) serves as an inner vertex along the path \(P\). A \(v_1\)-\(v_2\) geodesic of \(G\) is universally shared by all vertices within the bounded interval \(l(v_1, v_2)\). Let’s consider a non-empty set \(I(S) = \bigcup_{v_1,v_2 \in S} l(v_1,v_2)\). In accordance with (Beulah et al., 2018; Buckley et al., 1988; Chartrand et al., 2002; Chartrand et al., 2000; Joseph et al., 2022), if \(S\) qualifies as a geodetic set and \(G\) stands as a connected graph, then \(I(S)\) equates to \(V(G)\). The geodetic number \(g(G)\) signifies the smallest cardinality \(S\) within \(G\).

The edge chromatic number, also known as the chromatic index, is the bare minimum of colors needed to color each edge in a graph \(G\) so that no two edges sharing a common vertex have the same color. The number of various colors used in a minimum edge coloring may also be utilized to understand it. A graph’s edge chromatic number must at least equal the largest degree of its vertices, which is an essential consideration. But as demonstrated by studies by (Vizing, 1964) and (Gupta, 1966), no graph can have an edge chromatic number larger than one color more than its maximum degree. There are only two kinds of graphs as a result: those with an edge chromatic number equal to their maximum degree and those with an edge chromatic number one higher than their maximum degree (Soifer, 2009).

Edge Geodetic Number, represented by the notation \(g(G)\), is the lowest size among all sets of edges that make up Edge Geodetic sets. Assume that set \(S \subseteq V(G)\) meets the criteria for an Edge Geodetic set if, for each pair of vertices \(u\) and \(v\) inside set \(S\), every edge in \(G\) is a portion of the shortest path between \(u\) and \(v\). This is done by treating set \(S \subseteq V(G)\) as a finite connected graph. When \(C\) includes every appropriate \(k\)-edge color in \(G\), we define a set \(C \subseteq E(G)\) as a chromatic index set, where the chromatic index stands for the edge in \(G\) that has the smallest size. The phrase “Chromatic Index Number” and its symbol, \(\chi’(G)\), refer to the collection of these minimal cardinality chromatic index values.

Clearly, a geodetic set is not the conventional edge chromatic set in \(G\). The opposite is also not usually true, though. This led us to investigate the ground-breaking idea of “Edge Geo Colouring,” a geodetic method. The sets of vertices in graph \(G\) that are both geodetic sets and chromatic index sets are examined; these sets are referred to as “Edge Geo Chromatic sets.”
Combining an edge geodetic set with a chromatic index set yields edge geo chromatic sets in G. All Edge Geo Chromatic sets in G have the same cardinality, but the Edge Geo Chromatic number, indicated as \( \chi'_e(G) \), has the least cardinality. A chromatic index set and an edge geodetic set are combined to create an edge geo chromatic set, which is represented by the symbol \( S'_e \subseteq V(G) \). A linked Geo Chromatic graph with two edges and n orders is described.

The minimal cardinality within G is represented by the edge geochromatic number, sometimes called the edge chromatic index number. The edge geochromatic number of G is also known as the edge geodetic chromatic number. We determined the upper and lower bounds of the Edge Chromatic number for several kinds of common graphs. Furthermore, we have demonstrated that if a graph G has a dimension of d, then \( \chi'_e(G) \leq n - d + 2 \).

A geodetic set is definitely not the typical G edge chromatic set. The opposite is typically false as well. Because of this, we decided to research the ground-breaking "Edge Geo Colouring" geodetic technique. The subsets of vertices that have both a geodetic set and a chromatic index are examined in a graph G. These sets are referred to as edge geochromatic sets. The intersection of an edge geodetic set with a chromatic index set yields edge geo chromatic sets of G. All edge geo chromatic sets of G are known to have the edge geo chromatic number \( \chi'_e(G) \) as having the least cardinality.

An edge geochromatic \( S'_e \subseteq V(G) \) is a chromatic index set and an edge geodetic set. We describe a connected, two-edge, n-order geochromatic graph. The lowest cardinality of a graph is the edge geochromatic number, sometimes referred to as the edge chromatic index number. The edge geochromatic number is also known as the edge geodetic chromatic number of Graph G. We determined the edge geochromatic number and its bounds for a few common graphs. Additionally, we have demonstrated that if a graph G has a dimension d, then \( \chi'_e(G) \leq n - d + 2 \).

2. Materials and Methods

In Section 3, we present the idea of the Edge Geo Chromatic number and give various instances and results to illustrate it. In Section 4, bounds and general results are covered, and in Section 5, The Diameter of an Edge Geo Chromatic Number in G is examined.

Theorem 2.1 (Chartrand et al., 2002) Every geodetic set of a graph contains its extreme vertices.
Theorem 2.2 (Buckley et al., 1988) For a connected graph G, g(G) = n if and only if G = K_n.
Theorem 2.3 (Buckley et al., 1988) The geometric number of a tree T is the number of end vertices in T.

3. Results

3.1 Edge Geo Chromatic number

Definition 3.2: For at least two vertices in a connected graph G, a set \( S \subseteq V(G) \) is called edge geodetic set if every edge of G is contained in an \( u - v \) path for some \( u, v \in S \). A set \( S'_e \) of vertices in G is said to be edge geo chromatic set if \( S'_e \) is both a edge geodetic set and also chromatic index set, edge geo chromatic number \( \chi'_e(G) \) is the minimum cardinality of an edge geo chromatic set of G.

Example 3.3: In the case of a connected graph G, as shown in Figure 1, we have \( S = \{v_1, v_4\} \) as an edge geodetic set of G. Given that G can be colored with three colors, \( C = \{v_1, v_4\} \) constitutes a minimal chromatic index set. Consequently, the set \( S = \{v_1, v_4\} \) represents the minimum Edge Geo Chromatic Number for G. This implies that \( \chi'_e(G) = 2 \).

![Figure 1 The Graph G with \( \chi'_e(G) = 2 \).](https://www.malque.pub/ojs/index.php/msj)

Example 3.4: In the case of a connected graph G depicted in Figure 2, we have \( S = \{v_1, v_2, v_3, v_4\} \) as an edge geodetic set of G. In this scenario, \( \chi'(G) = 3 \). This set, \( S = \{v_1, v_2, v_3, v_4\} \), represents the smallest edge geodetic set for G, with g(G) equal to
4. Therefore, \( \chi'(G) \) is less than \( g_1(G) \) in this context. In contrast, when we refer back to Figure 1, we find that \( \chi'(G) \) equals 3, but \( g_1(G) \) equals 2. In this case, \( \chi'(G) \) is greater than \( g_1(G) \).

\[ \text{Figure 2: The Graph G with } \chi'(G) < g_1(G). \]

Remark 3.5: Take a look at Figure 1's graph G. Here, we see that \( S = \{v_1, v_4\} \) is both a Geo Chromatic Number and an Edge Geo Chromatic Number, implying that both \( \chi'_e(G) = \chi'_v(G) = 2 \). This shows that the Geo Chromatic Number and the Edge Geo Chromatic Number of a graph represent different ideas.

Theorem 3.6: For \( n \) – Path graph \( P_n \) (\( n > 2 \)),

\[ \chi'_e[P_n] = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 3, & \text{if } n \text{ is even.} \end{cases} \]

Proof: Consider the graph \( P_n \), which is a path graph with a vertex set denoted as \( V(G) = \{v_1, v_2, v_3, ..., v_n\} \), and an edge set denoted as \( E(G) = \{e_1, e_2, e_3, ..., e_{n-1}\} \). The length and diameter of \( P_n \) are both equal to \( n - 1 \). Let's assume a proper edge coloring of \( P_n \), where the color \( C_1 \) is assigned to edges \( e_{2k+1} \) for \( k = 1, 2, 3, ..., n \), and the color \( C_2 \) is assigned to edges \( e_k \) for \( k = 1, 2, 3, ..., n \). Now, let's explore the following scenarios.

In the case of \( P_n \) having an odd number of vertices, let's say with \( v_1 \) and \( v_n \) as the endpoints, it becomes evident that \( S = \{v_1, v_n\} \) represents the smallest edge geodetic set. Consequently, the edge geodetic number \( g_1(P_n) \) is equal to 2. Therefore, we can assign color \( C_1 \) to the vertex \( v_1 \) and color \( C_2 \) to the vertex \( v_n \), resulting in a 2-coloring, making the set \( S = \{v_1, v_n\} \) the minimal chromatic index set \( C \) for \( P_n \).\n
\[ g_1(P_n) = |S| \text{ and } \chi'_e(P_n) = |C| = |S|. \]

Hence \( |S| = |S'_{C_1}| = \chi'_e(P_n) = 2 \).

In the case \( P_n \) has an even number of vertices, let's say with \( v_1 \) and \( v_n \) as the endpoints, the edge geodetic set is minimal, consisting of \( S = \{v_1, v_n\} \). In this case, \( g_1(P_n) \) equals 2. However, the set \( S = \{v_1, v_n\} \) does not form a chromatic index set because \( v_1 \) and \( v_n \) are both incident with the same color class. This means that \( \chi'_e(P_n) > |S| = 2 \).

Now, let's examine the neighborhood of vertices \( v_1 \) and \( v_n \) denoted as \( N(v_1) \) and \( N(v_n) \). If both \( N(v_1) \) and \( N(v_n) \), when included in \( S \), contain an edge that is incident with a different color class, then \( S' = \{S \cup \{N(v_1)\} \cup \{N(v_n)\}\} \) becomes an edge geodetic set of \( P_n \), but it is not minimal. Consequently, in this case, \( \chi'_e(P_n) \) is less than 4, which equals \( |S| \).

However, if we choose a minimal edge geodetic set by selecting an edge from either \( N(v_1) \) or \( N(v_n) \) to be in \( S \), we find that \( \chi'_e(P_n) \) equals 3.

Remark 3.7: Consider the path graph \( P_2 \), which contains a vertex set of \( \{v_1, v_2\} \) and a single, one-length edge \( \{e_1\} \). To \( P_2 \), where we've colored \( e_1 \) with \( C_1 \), let's add suitable edge coloring. The edge geodetic set with a cardinality of 2 is formed by \( \{v_1, v_2\} \) as a result of \( g_1(S) \) equaling \( (v_1, v_2) \). The chromatic index number criterion for \( P_2 \) is satisfied by this value of 2, which also acts as the minimal edge geodetic set. Hence \( |S| = |S'_{C_1}| = \chi'_e(P_2) = 2 \).

Theorem 3.8: For \( n \) – Cycle graph \( C_n \) (\( n > 2 \)),

\[ \chi'_e[C_n] = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd}. \end{cases} \]

Proof: Consider \( C_n \) as a cycle graph with a vertex set \( V(G) = \{v_1, v_2, v_3, ..., v_n\} \) and an edge set \( E(G) = \{e_1, e_2, e_3, ..., e_n\} \). In this context, the sets \( S = \{v_1, v_{n+2}\}/2 \) and \( S = \{v_1, v_{n+3}/2\} \) represent the smallest edge geodetic sets for even and odd cycle \( C_n \), respectively. Let's explore the following two scenarios.
Case (i): When \( n \) is even, the set

\[
S = \left\{ v_1, v_{n+2} \right\}
\]

becomes the edge geodetic set of \( C_n \), resulting in \( g_1(C_n) \) equaling 2, which is the minimum value. By appropriately assigning edge colors to \( C_n \), where edges \( e_1, e_2, e_3, \ldots, e_{2k} \) are colored with \( C_1 \), and edges \( e_2, e_4, \ldots, e_{2k} \), where \( k = 1, 2, 3, \ldots, n \), receive color \( C_2 \), we achieve a 2-coloring of \( C_n \). Consequently, the set is incident with different color classes, making it a chromatic index set of \( C_n \). Therefore, the set \( S \) satisfies both the criteria of an edge geodetic set and a chromatic index set for \( C_n \), and it is the minimal set that does so. As a result, it represents the minimum edge geo chromatic set of \( C_n \). Hence, this leads to the conclusion denoted as \( \chi'_e(C_n) = 2 \).

Case (ii): When \( n \) is odd, the set

\[
S = \left\{ v_1, v_{n+1}, v_{n+3} \right\}
\]

represents the minimal edge geodetic set of \( C_n \), resulting in \( g_1(C_n) \) being equal to 3. By applying appropriate edge coloring to \( C_n \), where edges \( e_1, e_2, e_3, \ldots, e_{2k-1} \) are assigned color \( C_1 \), edges \( e_2, e_4, \ldots, e_{2k} \), where \( k = 1, 2, 3, \ldots, n \), receive color \( C_2 \), and edges incident with vertices \( \left\{ v_{n+1}, v_{n+3} \right\} \) are colored with \( C_3 \), we achieve a proper coloring of \( C_n \). Consequently, the set

\[
S = \left\{ v_1, v_{n+1}, v_{n+3} \right\}
\]

is incident with different color classes and serves as a chromatic index set of \( C_n \). Therefore, the set \( S \) meets both criteria. Consequently, \( \chi'_e(C_n) = 3 \) becomes the minimum edge geo chromatic set of \( C_n \), leading to the result denoted as

\[
\chi'_e(C_n) = 3.
\]

Theorem 3.9: For the Star graph \( n > 1 \),

\[
\chi'_e(K_n) = n.
\]

Proof: Consider \( K_n \) as a star graph with \( n \) edges and \( n+1 \) vertices, where the vertex set is \( \{v, v_1, v_2, v_3, \ldots, v_n\} \) and the edge set is \( \{e_1, e_2, e_3, \ldots, e_n\} \). In this scenario, the set comprising all the endpoints serves as the minimum edge geodetic set, denoted as \( S \), for \( K_n \), i.e., \( S = \{v, v_1, v_2, v_3, \ldots, v_n\} \). Now, through proper edge coloring of \( K_n \) with each edge receiving a distinct color, and assigning colors \( C_1, C_2, C_3, \ldots, C_n \) to the edges \( vv_1, vv_2, vv_3, \ldots, vv_n \), we establish a proper edge coloring. As a result, the set \( S = \{v, v_1, v_2, v_3, \ldots, v_n\} \) is incident with different color classes. Hence, the set \( |S| = |S'| \) represents the minimal edge geodetic set and also functions as the chromatic index set for \( K_n \). Consequently, this leads to the conclusion indicated as

\[
\chi'_e(K_n) = n.
\]

Theorem 3.10: For \( m, n \) - Double star graph \( DS_{mn} \) (\( m > 1, n > 1 \)),

\[
\chi'_e(DS_{mn}) = m + n + 1.
\]

Proof: Consider \( DS_{mn} \) as a double star graph, where it is connected by an edge \( uv \). Let’s designate \( u \) and \( v \) as the central vertices of the \( DS(m) \) and \( DS(n) \) graphs, respectively. Additionally, let \( u_1, u_2, u_3, \ldots, u_m \) represent the neighbors of \( u \), and \( v_1, v_2, v_3, \ldots, v_n \) denote the neighbors of \( v \). Now, we can assign the edges \( uu_1, uu_2, uu_3, \ldots, uu_m \) with labels \( e_1, e_2, e_3, \ldots, e_m \) and the edges \( vv_1, vv_2, vv_3, \ldots, vv_n \) with labels \( e_{m+1}, e_{m+2}, \ldots, e_{mn} \). Furthermore, the edge \( uv \) can be assigned the label \( e \). Let the degree \( u = m > 2 \) and \( v = n > 2 \). Now, let’s apply a suitable edge coloring to \( DS_{mn} \). We can designate the edges \( e_1, e_2, e_3, \ldots, e_m \) with colors \( C_1, C_2, C_3, \ldots, C_m \), respectively. Similarly, the edge \( e \) is assigned the color \( C_{m+1} \), and the edges \( e_{m+1}, e_{m+2}, \ldots, e_{mn} \) are given the same set of colors \( C_1, C_2, C_3, \ldots, C_m \). It’s important to note that the set comprising all the endpoint vertices of \( DS_{mn} \) serves as the minimal edge geometric set, denoted as \( S \), which is indeed minimal. However, this set does not fulfill the criteria of being a chromatic index set. Therefore, \( \chi'_e(DS_{mn}) > |S| \).

Now, let’s examine the neighboring vertices of \( u \)'s or \( v \)'s, where \( i \) ranges from 1 to \( m \) and \( j \) ranges from 1 to \( n \). If both \( N(u_1) \) and \( N(v_j) \), when included in the set \( S \), contain an edge that is incident with different colors at \( u_1 \) and \( v_j \), then \( S' = \{S \cup \{N(u_1_1)\}, \cup \{N(v_j)\}\} \) becomes an edge geo chromatic set of \( DS_{mn} \). However, in this scenario, \( S' = \{S \cup \{N(u_1)\}, \cup \{N(v_j)\}\} \) is not minimal. Therefore, to identify a minimal edge geo chromatic set, we can select a vertex from either \( N(u_1) \) or \( N(v_j) \) to be included in \( S \). Hence it results \( \chi'_e(DS_{mn}) = m + n + 1 \).
Theorem 3.11: For any Bistar graph $K_{1,n,n}$ ($n>1$), $\chi^\prime_c(K_{1,n,n}) = n$.

Proof: Let's consider the Bistar graph $K_{1,n,n}$, which is derived by subdividing the star graph $K_{1,n}$. To begin with, $K_{1,n}$ is a star graph with a vertex set $\{v_1, v_2, v_3, ..., v_n\}$ and the center vertex is designated as $v$. The vertices resulting from the subdivision of $K_{1,n}$ are denoted as $\{u_1, u_2, u_3, ..., u_n\}$. This configuration gives rise to the Bistar graph $K_{1,n,n}$. Therefore, the order of $K_{1,n,n}$ is represented as $|V(G)| = 2n+1$, and its size is denoted as $|E(G)| = 2n$.

Now, we can assign the edges $e_1, e_2, e_3, ..., e_{n}$ to the connections $vu_1, vu_2, vu_3, ..., vu_n$ and the edges $e_{n+1}, e_{n+2}, e_{n+3}, ..., e_{2n}$ correspond to $u_1v_1, u_2v_2, u_3v_3, ..., u_nv_n$. For proper coloring of $K_{1,n,n}$, we assign color classes $C_1, C_2, C_3, ..., C_n$ to the vertices $vu_1, vu_2, vu_3, ..., vu_n$ and allocate the same color class to the edges in such a way that $u_1v_1, u_2v_2, u_3v_3, ..., u_nv_n$.

It's noteworthy that the set comprising all the endpoint vertices of $K_{1,n,n}$ serves as the minimal geodetic set, which we denote as $S$. Therefore, $S = \{v_1, v_2, v_3, ..., v_n\}$ represents the minimum edge geodetic set of $K_{1,n,n}$ and it also fulfills the criteria for being a chromatic index set. Consequently, this set satisfies the given condition, resulting in $S_c = \{v_1, v_2, ..., v_n\}$ being the minimal edge geo chromatic set of $K_{1,n,n}$. Hence it results $\chi^\prime_c(K_{1,n,n}) = n$.

Corollary 3.12: For any Tristar graph $K_{1,n,n,n}$ ($n>1$), $\chi^\prime_c(K_{1,n,n,n}) = n$.

Proof: Follows from theorem 3.10.

Theorem 3.13: For $n$ - Complete graph $K_n$ ($n>3$), $\chi^\prime_c(K_n) = n$.

Proof: Consider the complete graph $K_n$, which consists of $n$ vertices and $\frac{n(n-1)}{2}$ edges, where each vertex has a degree of $n - 1$. In $K_n$, every edge serves as the minimal edge geodetic set, denoted as $S$, with $g_c(S)$ equal to $n$. This is because each edge in $K_n$ is assigned distinct colors. As a consequence of this proper coloring, it leads to the result $\chi^\prime_c(K_n) = n$.

Theorem 3.15: For $m,n$ -Complete bipartite graph $K_{m,n}$

Proof: Consider the complete bipartite graph $G = K_{m,n}$, where $U = \{u_1,u_2,u_3,..,u_m\}$ and $V = \{v_1,v_2,v_3,..,v_n\}$ represent the sets of all vertices in $G$, with both $m$ and $n$ being greater than 1.

Case (i): When we assume that $m$ is equal to $n$, we end up with $|V(G)| = 2m$ or $2n$ vertices in total. Consequently, the set $S$ can be either $|U|$ or $|V|$, and it serves as the minimum edge geodetic set for $G$. When we appropriately assign edge colors to $G$, the edges incident to the set $S$ receive distinct color classes. This makes the set $S$ the minimal edge geodetic set that also fulfills the criteria of being a chromatic index set. As a result, $S_c = |S|$ represents the minimum edge geo chromatic set, leading to the conclusion $\chi^\prime_c(G)$ being either $m$ or $n$.

Case (ii): Let's consider the scenario where both $m$ and $n$ are greater than 1. In this case, the set $S$, which can be defined as the minimum between $|U|$ and $|V|$, serves as a geodetic set for $G$, and it is minimal. By assigning the appropriate edge colors to $G$, the edges incident to the set $S$ are colored differently. As a result, the set $S$ becomes the minimal edge geodetic set while also satisfying the criteria for being a chromatic index set. Hence, $S_c = |S|$ represents the minimal edge geo chromatic set, leading to the conclusion that $\chi^\prime_c(G)$ is the minimum between $m$ and $n$.

Theorem 3.16: For $n$ - wheel graph $W_n$ ($n>3$), $\chi^\prime_c(W_n) = n$.

Proof: Let's consider the wheel graph denoted as $W_n$, which is formed by combining $K_1$ with $C_{n-1}$ and consists of $n$ vertices and $2n - 2$ edges. Now, suppose $v'$ is the central vertex of $K_1$, and the remaining vertices are $\{v_1, v_2, v_3, ..., v_{n-1}\}$ of $C_{n-1}$. As a result, the set $S$, which is defined as $\{v_1, v_2, v_3, ..., v_{n-1}\}$ of $C_{n-1}$. Through proper edge coloring of $W_n$, the edges incident to the set $S$ are assigned distinct color classes. This ensures that the set $S$ is not only the minimal edge geodetic set but also functions as the chromatic index set for $W_n$. Consequently, $S_c = |S|$ signifies the minimum edge geo chromatic set, leading to the conclusion represented as $\chi^\prime_c(W_n) = n$.

4. Bounds for the Edge Geo Chromatic Number

Theorem 4.1: For every order $n$ of graph $G$, $2 \leq \chi^\prime_c(G) \leq n$. 

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Proof: An edge geo chromatic set requires a minimum of 2 vertices. Therefore, we use \( \chi'_e(P_n) = 2 \) when \( n \) equals 2 or when \( n \) is an odd number, and we use \( \chi'_e(C_n) = 2 \) when \( n \) is an even number. Thus for a connected graph \( G \) of order \( n \geq 3 \), the edge geo chromatic number \( k \geq 3 \) and \( n \geq k \). Thereby \( \chi'_e(K_n) = \chi'_e(K_{iv}) = n \). It is clear that the set of all vertices of \( G \) is an edge geo chromatic set of \( G \). So, that \( \chi'_e(G) = n \). Hence result that \( 2 \leq \chi'_e(G) \leq n \).

Remark 4.2: The bounds provided in Theorem 4.1 are significant. For the complete graph \( K_n \) (where > 1), \( \chi'_e(K_n) = n \) holds true. In the case of a path set with end vertices, \( P_n \) (where \( n > 1 \)), it constitutes its exclusive edge geo chromatic set, leading to the conclusion represented as \( \chi'_e(G) = 2 \). Therefore, the complete graph \( K_n \) has the highest positive edge geo chromatic number, which is \( n \), while the non-trivial path exhibits the lowest edge geo chromatic number, which is 2.

Corollary 4.3: For each graph \( G \) that has \( k \) end vertex vertices,
Proof: Follows from theorem 4.1 and theorem 3.8.

Corollary 4.4: For any tree \( T \), the edge geo chromatic number \( \chi'_e(T) \) equals the number of end vertices in \( T \).
Proof: Follows from theorem 4.1 and theorem 3.8.

Corollary 4.5: For every pair \( k, n \) of integer with \( 2 \leq k \leq n \) there exist a connected graph \( G \) of order \( n \) such that \( \chi'_e(G) = k \).

Proof: If we assume that \( k \) equals \( n \), then the resulting graph becomes a complete graph \( K_n \). In this scenario, for every pair of integers, \( 2 \leq k \leq n \) holds true. Consequently, there exists a tree of order \( n \) with \( k \) endpoints vertices which are incident with distinct color classes. Hence the result follows \( \chi'_e(G) = k \).

Corollary 4.6: For the Hypercubic graph \( Q_n \) \( (n \geq 1) \), \( \chi'_e(Q_n) = 2 \).
Proof: Let’s examine the graph \( Q_n \), which consists of \( 2n \) vertices, and denote the vertices as \( \{v_1, v_2, v_3, \ldots, v_n\} \), where each \( v_i \) (\( 1 \leq i \leq n \)) represents a binary digit, either 0 or 1. Consider \( v'_i \) as the complement of \( v_i \) meaning \( v'_i \) can take on the values of 0 or 1 in a similar order. In this binary order, two vertices differ by exactly one position if and only if there are two adjacent vertices in \( Q_n \). Now, let \( u \) and \( v \) be two vertices in \( Q_n \) such that their distance \( d(u, v) \) equals the diameter of \( Q_n \), which is \( n \).

When \( n = 1 \), \( Q_1 \) has two vertices \( v_1 \) and \( v_2 \) (say) connected by an edge \( e \). Follow from theorem 3.5.

When \( n \) equals 2, \( Q_2 \) consists of 22 vertices, denoted as \( \{v_1, v_2, v_3, v_4\} \). In this case, the four vertices of \( Q_2 \) are adjacent if and only if their binary representations differ in exactly one position. Consequently, this graph corresponds to \( C_4 \). Thus, its result from theorem 3.7.

When \( n \) is greater than 2, \( Q_n \) comprises \( 2n \) vertices. For simplicity, consider \( u \) as the set \( \{v_1, v_2, v_3, \ldots, v_n\} \) and \( v \) as the set \( \{v'_1, v'_2, v'_3, \ldots, v'_n\} \). In this context, when the distance \( d(u, v) = n \), we have a path \( P: u = \{v_1, v_2, v_3, \ldots, v_n\} \{v'_1, v'_2, v'_3, \ldots, v'_n\} \{v_1, v'_2, v_3, \ldots, v_n\} \ldots \ldots \{v_1, v_2, v_3, \ldots, v'_n\} \).

Corollary 4.7: If \( G \) has exactly one \( n \)-degree - 1 vertex, then \( \chi'_e(G) = n - 1 \).
Proof: Follow from theorem 3.8.

Corollary 4.8: For any tree \( T \) with \( n \geq 3 \) vertices \( \chi'_e(T) = n - 1 \) iff \( T \) is a Star.
Proof: Let’s suppose \( T \) is a star graph, specifically \( K_{1,n-1} \). According to theorem 3.8, \( \chi'_e(T) = n - 1 \) holds true in this case. Conversely, assuming \( X \), let’s consider whether \( T \) is not a star. In such a scenario, there must be vertices \( u \) and \( v \) for which the distance \( d(u, v) \) equals the diameter of the graph, i.e., \( \text{diam}(u, v) \geq 3 \). Consequently, there are at least two vertices in \( T \) that are incident with different color classes. This contradicts Theorem 3.7 [7]. As a result, we conclude that \( T \) must indeed be a star graph.

5. Diameter of an Edge Geo Chromatic Number

Theorem 5.1: If \( G \) is a graph with diameter \( d \), then \( \chi'_e(G) \leq n - d + 2 \).
Proof: If $G$ is a path graph $P_n$, $n$ is even then $\chi'_{ge}(G) \leq n - d + 2$. Then for a complete graph $K_n$, $\chi'_{ge}(G) = n$. For a star graph $K_{1,n-1}$, we see that $S$ is an edge geo chromatic number of $G$. So that $\chi'_{ge}(G) \leq n - d + 2$.

Corollary 5.2: If $G$ is a connected graph with $d > 1$, then $\chi'_{ge}(G) \leq n - 1$.

Proof: Follows from theorem 5.1 and 3.5.

Theorem 5.3: For any tree $T$ of order $n$ and diameter $d$ then $\chi'_{ge}(T) \leq n - d + 2$.

Proof: Let’s take any tree $T$ as a non-trivial connected graph with $n$ vertices and a diameter of $d$. Suppose $v_1$ and $v_2$ are two vertices in $T$ such that the distance between them, $d(v_1, v_2)$, equals $d$. Now, consider a path from $v_1$ to $v_2$ with a length of $d$, denoted as $v_1 = v_{i_0}, v_{i_1}, v_{i_2}, ..., v_{i_d} = v_2$ (let’s say). Additionally, consider $S$ as the set of vertices in $T$ excluding $\{v_1, v_2, ..., v_{i_d}\}$. This set $S$ serves as the minimal edge geodetic set and may receive the same or different colors when edges are incident to it.

Now, assume that $S$ does not belong to the minimum chromatic index set of $T$. In this case, select the neighboring vertices of $S$ that are incident with different color classes. As a result, $S'$ becomes the minimal edge geo chromatic set of $T$, leading to the conclusion represented as $\chi'_{ge}(T) \leq n - d + 2$.

6. Conclusions

In this paper, we explore the computation of the edge geochromatic number and investigate limitations and diameters for different fundamental graphs. Additionally, we extend the concept of the edge geochromatic number to several other graphs.

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Conflict of Interest

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